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Second-order beliefs and unambiguous events

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Abstract

This paper axiomatizes expected utility models with second-order beliefs that distinguish between exogenously given unambiguous and ambiguous events. Second-order subjective expected utility of Seo (2009) and Second-order maxmin expected utility of Nascimento and Riella (2011) are elaborated. Extracted second-order beliefs agree with some first-order probability on unambiguous events with probability of one.
Keywords: Ambiguity, Unambiguous events, Second-order belief

1 Introduction

Since Ellsberg (1961) found a class of uncertainty, named ambiguity, it has been one of the central subjects of decision theory. One of Ellsberg's thought experiments is as follows: There is an urn containing 90 balls, each of which is colored red, green, or blue. Thirty balls are known to be red, but there is no information about the ratio of green and blue balls. One ball will be taken out from the urn. A decision maker (henceforth DM) is confronted with four bets on its color. Bet 1 gives her 100 dollars if the ball is red and nothing otherwise. Bet 2 gives 100 dollars if the ball is green. Bet 3 gives 100 dollars if the ball is red or blue. Bet 4 gives 100 dollars if the ball is green or blue. DM is asked to choose between bet 1 and 2, and then between 3 and 4. It is well known that typical subjects prefer to bet 1 over 2 and bet 4 over 3. This behavior contradicts any models based on probabilistic belief. The class of uncertainty that cannot be represented by probabilities is called ambiguity.

An approach to analyze ambiguity is to provide models that capture the above-mentioned Ellsberg type behavior. Since the pioneering works of Schmeidler (1989) and Gilboa and Schmeidler (1989), many alternative models have been provided. Another stream is to define ambiguity by distinguishing events to which DM assigns probabilistic likelihood. Early works in this line are Zhang (2002) and Epstein and Zhang (2001).

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The present paper contributes to the latter literature by providing a set of axioms that specializes second-order subjective expected utility (SOSEU) model of Seo (2009). Under the obtained model, exogenously given events can be interpreted as unambiguous events. A similar result holds for a more general model, second-order maxmin expected utility (SOMEU) of Nascimento and Riella (2011).

SOSEU consists of three parameters: an expected utility function u, a strictly increasing real function v, and a probability measure on the set of probability measures on the state space S, $m \in \Delta(\Delta S)$. Second-order probability m represents ignorance of DM about probabilistic law. A utility function of SOSEU is written as

$$V(f) = \int_{\Delta S} v\left(\int_S u(f(s))d\mu(s)\right) dm(\mu),$$

and it captures the Ellsberg type behavior. The idea of SOSEU appears in Savage (1954) and the model is axiomatized by Klibanoff et al. (2005) and Seo (2009) in different settings.

Under SOSEU, it seems natural to define unambiguous events as events such that the distributions of the probability assigned to them, induced by second-order belief, degenerate.[1] The main theorem in this paper shows that a preference relation satisfies a set of axioms if and only if it has an SOSEU representation under which exogenously given events are unambiguous in this sense.

Many authors have written on probabilistic belief on endogenous and exogenous events. Zhang (2002) and Epstein and Zhang (2001) provide different definitions of unambiguous events based on the intuition of the Sure-Thing Principle of Savage (1954) and P4* of Machina and Schmeidler (1992), respectively. Kopylov (2007) refines their arguments. Sarin and Wakker (1992) axiomatizes Choquet expected utility whose capacity is additive on exogenously given events. Qu (2013) axiomatizes maximin expected utility (MEU) whose multiple priors degenerate on exogenously given events and defines unambiguous events under MEU. The papers most related to the present one are Klibanoff et al. (2005) and Klibanoff et al. (2011). Klibanoff et al. (2005) provided an alternative foundation of SOSEU (smooth ambiguity model in their terminology) and proposed a definition of unambiguous events, which works under their model. Klibanoff et al. (2011) compares their definition with earlier ones.

In section 2, the setup is presented and the main results are described. In section 3, I compare the Theorem 1 in this paper with the characterization of unambiguous events in Klibanoff et al. (2005), and discuss the possibility of future research. All proofs are collected in the appendix.

[1]This is a model-based definition and some authors argue that the concept of ambiguity should be formalized without referring to any particular model (Epstein and Zhang (2001)). However, it should not be dismissed immediately. Some economists believe that intuitive stories of models make us more comfortable to rely on predictions of them and they enhance the modeler's reasoning process (Dekel and Lipman (2010)). In this point of view, it is worthwhile to first provide a definition of unambiguous events based on well-established models such as SOSEU and then investigate its behavioral implications.

2 Model and results

2.1 Setup

I adopt the original framework of Anscombe and Aumann (1963) as in Seo (2009). Let X denote the set of prizes and assume it is a separable metric space. For any topological space Y, denote ΔY the set of all Borel probability measures on Y. Any set of probability measures is endowed with the weak topology. Let S denote the set of states and assume it is finite. A Subset of S is called an event. A function from S to ΔX is called an act. Let $\mathcal{F}$ denote the set of acts. $\mathcal{F}$ is endowed with product topology. The choice set of DM is the set of probability measures on $\mathcal{F}$, $\Delta\mathcal{F}$. Prizes are denoted by x, y, z. Elements of ΔX are denoted by p, q, r. Acts are denoted by f, g, h. Elements of $\Delta\mathcal{F}$ are denoted by P, Q, R.

The sets X and $\mathcal{F}$ can be seen as subsets of ΔX and $\Delta\mathcal{F}$ respectively in an obvious manner. An element of ΔX can be identified with a constant act. So $X \subset \Delta X \subset \mathcal{F} \subset \Delta\mathcal{F}$. For any family of events $\mathcal{A}$, let $\mathcal{F}_{\mathcal{A}}$ denote the set of acts that are measurable with respect to $\mathcal{A}$.

DM's choice behavior is modeled as a binary relation $\succeq$ on $\Delta\mathcal{F}$. The relations $\succ$ and $\sim$ denote the asymmetric and symmetric part of $\succeq$.

In the rest of paper, let $\mathcal{U}$ denote an exogenously given family of events that contains S and is closed under complementation. As shown below, under suitable axioms, $\mathcal{U}$ can be interpreted as a set of unambiguous events. For example, in the Ellsberg's experiment above, DM knows that the probability a red ball is taken out is one third. So R, the event that a red ball is taken out, is an unambiguous event. One can take $\mathcal{U} = \{S, \emptyset, R, R^c\}$ as the set of her unambiguous events.

2.2 Unambiguous events under SOSEU

I list the axioms of Seo (2009) that characterize SOSEU representations. First two are common in the literature.

Axiom 1 (Order)**.** $\succeq$ is complete and transitive.

Axiom 2 (Continuity)**.** $\succeq$ is continuous.

A mixing operation on $\mathcal{F}$ is defined as $(\alpha f \oplus (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$ for any $f, g \in \mathcal{F},\ s \in S$. Then, $\mathcal{F}$ is a mixture space under this operation. The next axiom requires preferences to satisfy independence on the set of one-stage lotteries.

Axiom 3 (Second-Stage Independence)**.** For any $\alpha \in (0,1]$ and one stage lotteries $p, q, r \in \Delta X$, $\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r \Leftrightarrow p \succeq q$.

For $P, Q \in \Delta\mathcal{F}$ and $\alpha \in [0,1]$, $\alpha P + (1-\alpha)Q \in \Delta\mathcal{F}$ is a lottery such that $(\alpha P + (1-\alpha)Q)(B) = \alpha P(B) + (1-\alpha)Q(B)$ for any Borel set $B \subset \mathcal{F}$. The next axiom requires preferences to satisfy independence on the set of lotteries of acts.

Axiom 4 (First-Stage Independence)**.** For any $\alpha \in (0, 1]$ and probabilities $P, Q, R \in \Delta\mathcal{F}$, $\alpha P(1-\alpha)R \succeq \alpha Q + (1-\alpha)R \Leftrightarrow P \succeq Q$.

For $f \in \mathcal{F}$ and $\mu \in \Delta S$, define $\Psi(f, \mu) = \mu(s_1)f(s_1) \oplus \cdots \oplus \mu(s_{|S|})f(s_{|S|}) \in \Delta X$. If DM has a probabilistic belief μ, she identify an act f with the one-stage lottery $\Psi(f, \mu)$. For $P \in \Delta\mathcal{F}$ and $\mu \in \Delta S$, define $\Psi(P, \mu) \in \Delta(\Delta X)$ as $\Psi(P, \mu)(B) = P(\{f \in \mathcal{F} | \Psi(f, \mu) \in B\})$. This two-stage lottery gives some first-stage lottery p with the probability that the original lottery P assigns to the acts that are identified with p under μ. The next axiom requires that if a second-order lottery P is preferred to Q no matter which first-order subjective probability $\mu \in \Delta S$ is true, then P is indeed preferred to Q.

Axiom 5 (Dominance)**.** For any $P, Q \in \Delta\mathcal{F}$, if $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for all $\mu \in \Delta S$, then $P \succeq Q$.

Seo (2009) shows that these behavioral regularities characterize the following utility representation.

Definition 1. A tuple (u, v, m) is a second-order subjective expected utility (SOSEU) representation of $\succeq$ if $u : \Delta X \to \mathbb{R}$ is bounded continuous and mixture linear, $v : u(\Delta X) \to \mathbb{R}$ is bounded continuous and strictly increasing, $m \in \Delta(\Delta S)$ and V represents $\succeq$, where

$$
\begin{aligned}
V(P) &= \int_{\mathcal{F}} U(f) dP(f), \\
U(f) &= \int_{\Delta S} v\left(\int_S u(f) d\mu\right) dm(\mu).
\end{aligned}
$$

An SOSEU representation (u, v, m) is nondegenerate if u is not constant.

Under SOSEU, it is natural to define unambiguous events as those which the distributions of probability DM assigns to them are degenerated. The formal definition is as follows.

Definition 2. An event is naturally unambiguous under $m \in \Delta(\Delta S)$ if there exists $\alpha \in [0, 1]$ such that $\mu(E) = \alpha$ m-a.s..

To guarantee that any exogenously given event is naturally unambiguous in this sense, I introduce a new axiom. For this purpose, define $\Delta_{\mathcal{U}} = \{\mu \in \Delta S | \forall f \in \mathcal{F}_{\mathcal{U}}\ f \sim \Psi(f, \mu)\}$. This is the set of first-order probabilities such that DM is indifferent between any acts measurable with respect to $\mathcal{U}$ and the corresponding one-stage lottery constructed with it.

Axiom 6 ($\mathcal{U}$-Dominance)**.** For $P, Q \in \Delta\mathcal{F}$, if $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for all $\mu \in \Delta_{\mathcal{U}}$ then $P \succeq Q$.

$\mathcal{U}$-dominance is stronger than Dominance in that only first-order probabilities in $\Delta_{\mathcal{U}}$, a set smaller than ΔS, are relevant to DM. This can be interpreted as having stronger confidence in probabilistic law.

Finally, we put an auxiliary axiom to elicit probabilities.

Axiom 7 (Nondegenerate)**.** $P \succ Q$ for some $P, Q \in \Delta\mathcal{F}$.

Now the main result can be stated.

Theorem 1. Followings are equivalent.
1. $\succeq$ satisfies Order, Continuity, First-stage independence, Second-stage independence, $\mathcal{U}$-dominance, Nondegenerate.
2. $\succeq$ has a nondegenerate SOSEU representation (u, v, m) such that any $E \in \mathcal{U}$ is naturally unambiguous under m.

This theorem demonstrates that a preference satisfies the required axioms if and only if it has an SOSEU representation and each event in $\mathcal{U}$ is naturally unambiguous under the second-order belief. Hence DM behave as if she is an SEU maximizer when choosing among unambiguous acts. If one imposes $\mathcal{U}$-dominance for larger $\mathcal{U}$, the model becomes closer to SEU. Hence the theorem reveals what lie between SEU and SOSEU.

The model includes nondegenerate SOSEU as a special case.

Corollary 1. Suppose $\mathcal{U} = \{\emptyset, S\}$. Followings are equivalent.
1. $\succeq$ satisfies Order, Continuity, First-stage independence, Second-stage independence, $\mathcal{U}$-dominance, Nondegenerate.
2. $\succeq$ has a nondegenerate SOSEU representation.

2.3 Unambiguous events under SOMEU

In this subsection, I show a result similar to Theorem 1 for a more general model, Second-order maxmin expected utility of Nascimento and Riella (2011).

Definition 3. A tuple (u, v, M) is a second-order maxim expected utility (SOMEU) representation of $\succeq$ if $u : \Delta X \to \mathbb{R}$ is bounded continuous and mixture linear, $v : u(\Delta X) \to \mathbb{R}$ is bounded continuous and strictly increasing, $M \subset \Delta(\Delta S)$ is a nonempty, closed, and convex set, and V represents $\succeq$, where

$$V(P) = \min_{m \in M} \int_{\Delta S} \left[\int_{\mathcal{F}} v \left(\int_S u(f) d\mu \right) dP(f) \right] dm(\mu).$$

An SOMEU representation (u, v, M) is nondegenerate if u is not constant.

I list axioms that characterize SOMEU preferences. I adopt axioms different from Nascimento and Riella (2011) for simplicity. The following two axioms are counterparts of Uncertainty Aversion and Certainty Independence proposed in Gilboa and Schmeidler (1989), respectively. Because both are weaker than First-stage independence, SOMEU is a generalization of SOSEU.

Axiom 8 (Convexity)**.** For all $P, Q \in \Delta\mathcal{F}$, and $\lambda \in (0,1)$, if $P \sim Q$ then $\lambda P + (1-\lambda)Q \succeq Q$.

Axiom 9 (First-Stage Certainty Independence)**.** For any $P, Q \in \Delta\mathcal{F}$, $\overline{P} \in \Delta(\Delta X)$, and $\alpha \in (0,1]$, $\alpha P + (1-\alpha)\overline{P} \succeq \alpha Q + (1-\alpha)\overline{P} \Leftrightarrow P \succeq Q$.

The definition of naturally unambiguous events is extended to the case of SOMEU. An event is naturally unambiguous if all the probabilities in the second-order belief M assign common degenerate probability to the event.

Definition 4. An event E is naturally unambiguous under $M \subset \Delta(\Delta S)$ if there exists an $\alpha \in [0,1]$ such that $\mu(E) = \alpha$ m-a.s. for all $m \in M$.

The following theorem is the counterpart of Theorem 1 adapted for SOMEU.

Theorem 2. Followings are equivalent.
1. $\succeq$ satisfies Order, Continuity, Second-stage independence, First-stage Certainty Independence, Convexity, $\mathcal{U}$-dominance, Nondegenerate.
2. There is a nondegenerate SOMEU representation (u, v, M) such that any $E \in \mathcal{U}$ is naturally unambiguous under M.

This theorem demonstrates that $\mathcal{U}$-dominance brings unambiguity even under SOMEU. The following is an immediate consequence of Theorem 2.

Corollary 2. Suppose $\mathcal{U} = \{\emptyset, S\}$. Followings are equivalent.
1. $\succeq$ satisfies Order, Continuity, Second-stage independence, First-stage Certainty independence, Convexity, $\mathcal{U}$-dominance, Nondegenerate.
2. $\succeq$ has a nondegenerate SOMEU representation.

3 Concluding remarks

In this paper, I provided axiomatic models whose second-order beliefs agree with some first-order probability on unambiguous events with probability of one. This section compares Theorem 1 with Theorem 3 in Klibanoff et al. (2005), which characterizes unambiguous events for DM whose choice behavior is modeled by their smooth ambiguity model. They called an event E unambiguous if there is some event B with objective probability to obtain such that $xEy \sim xBy \Leftrightarrow yEx \sim yBx$, for any prize x and y. And then, they proved that an event is unambiguous in this sense if and only if the second-order beliefs of their smooth ambiguity model degenerate on E under a technical assumption on ambiguity attitude.

Their result can be replicated in the domain adopted here. Consider the following definition of unambiguous events.

Definition 5. An event $E \subset S$ is KMM unambiguous if for any $p, q \in \Delta X$ and $\alpha \in [0,1]$, $pEq \sim \alpha p \oplus (1-\alpha)q \Leftrightarrow qEp \sim \alpha q \oplus (1-\alpha)p$.

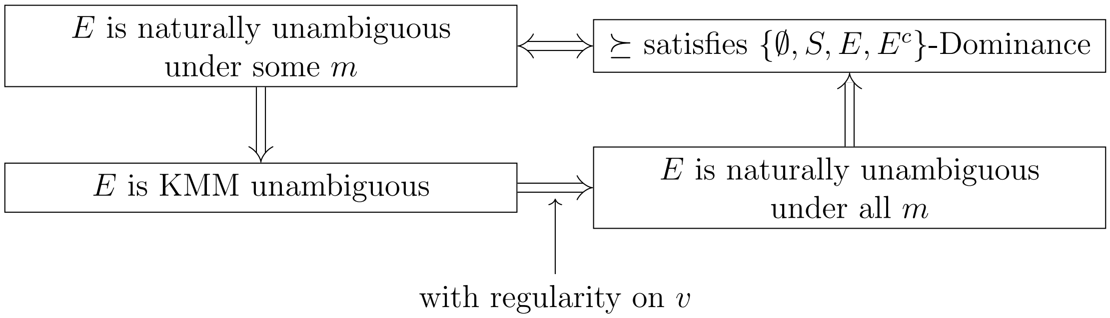

Figure 1: Relationships between characterizations of naturally unambiguous events under SOSEU.

Suppose DM believes that the probability of event B in the definition of Klibanoff et al. (2005) is α. Suppose also that she is indifferent between x and p, and between y and q. Then she will identify $\alpha p \oplus (1-\alpha)q$ with xBy. So the condition $xEy \sim xBy$ is equivalent to $pEq \sim \alpha p \oplus (1-\alpha)q$. Similarly, one can conclude that $yEx \sim yBx$ is equivalent to $yEx \sim \alpha q \oplus (1-\alpha)p$. Hence KMM unambiguity defined above captures the original idea. It is clear that an event is KMM unambiguous if it is naturally unambiguous under some m that constitutes an SOSEU representation of $\succeq$. The following proposition shows that it identifies the set of naturally unambiguous events. The proof is similar with Theorem 3 of Klibanoff et al. (2005) and omitted.

Proposition 1. Suppose $\succeq$ has an SOSEU representation (u, v, m) and there exists an open interval in $u(\Delta X)$ on which v is strictly concave or strictly convex. Then, an event is KMM unambiguous if and only if it is naturally unambiguous under m.

For any SOSEU representations (u, v, m) and (u', v', m'), v satisfies the regularity condition in the proposition if and only if v' does so.[2] So this proposition guarantees that if some representation satisfies the regularity condition, then KMM unambiguous events are naturally unambiguous under any m that constitutes an SOSEU representation. Figure 1 summarizes the relationships between $\mathcal{U}$-Dominance and KMM unambiguity.

Finally, I mention a direction of future research. A flaw of Proposition 1 is that it is proved under a condition on ambiguity attitude. Because an aim of smooth ambiguity model is to achieve a separation between belief and attitude, it is worth to investigate, under SOSEU, whether we can characterize unambiguous events without any condition on ambiguity attitude. More formally, it is desirable to characterize a family of events such that all elements of the family is naturally unambiguous under some fixed SOSEU representation of $\succeq$, and all other events are not naturally unambiguous under any SOSEU representation.

A natural approach toward this characterization is to consider a set

$$\mathcal{V} = \{\mathcal{U} | \succeq \text{ satisfies } \mathcal{U}\text{-Dominance}\}$$

[2]One can see this fact using Lemma C.1 in Seo (2009).

and investigate whether it has the largest element in the sense of set inclusion. However, I conjecture that it does not exist in general. Using Theorem 1, one can see that the largest element of $\mathcal{V}$ exists if and only if $\mathcal{U}, \mathcal{U}' \in \mathcal{V}$ implies $\mathcal{U} \cup \mathcal{U}' \in \mathcal{V}$. Hence, suppose $\mathcal{U}, \mathcal{U}' \in \mathcal{V}$. By Theorem 1, one can take two SOSEU representations (u, v, m) and (u', v', m') under which elements of $\mathcal{U}$ and $\mathcal{U}'$ are naturally unambiguous, respectively. One then needs to show $\mathcal{U} \cup \mathcal{U}' \in \mathcal{V}$. However, as shown in Seo (2009), uniqueness of second-order belief depends on curvature of v. Hence, in general, m and m' may well be different. This is an obstacle to show the desired result and the reason of my conjecture. However, whether the conjecture is true or not, this issue is open for future research.

A Appendix

Suppose $\succeq$ satisfies Order, Continuity, Second-stage Independence, $\mathcal{U}$-dominance, Nondegenerate. Then it satisfies Dominance, because it is weaker than $\mathcal{U}$-dominance. Denote the restriction of $\succeq$ to a subset Y of $\Delta\mathcal{F}$ as $\succeq|_Y$. Using Grandmont (1972), Theorem 2, take a bounded continuous mixture linear function $u : \Delta X \to \mathbb{R}$ that represents $\succeq|_\Delta X$. And because $\succeq|_{\Delta(\Delta X)}$ satisfies independence, there exists a bounded continuous $w : \Delta X \to \mathbb{R}$ such that $V(\overline{P}) = \int_{\Delta X} w(p) d\overline{P}$ represents $\succeq|_{\Delta(\Delta X)}$. Because w represents $\succeq|_{\Delta X}$, w and u are equivalent utility representation on ΔX. So there is a continuous and strictly increasing $v : u(\Delta X) \to \mathbb{R}$ such that $w = v \circ u$. Because $\succeq$ is not trivial, one can suppose $(0, 1) \subset v(u(\Delta X))$ without loss of generality.

Lemma 1. For any $P \in \Delta\mathcal{F}$, there exists some $\overline{P} \in \Delta(\Delta X)$ such that $P \sim \overline{P}$.

Proof. Seo (2009) Lemma B.2 shows that a map $(f, \mu) \mapsto \Psi(f, \mu)$ is jointly continuous. I show that a map $\mu \mapsto \Psi(P, \mu)$ is also continuous. Take any $P \in \Delta\mathcal{F}$, any bounded continuous function η defined on ΔX, and any sequence $\{\mu_n\}_n \subset \Delta S$ that converges to μ. Then,

$$\begin{aligned}\int_{\Delta X} \eta \, d\Psi(P, \mu_n) &= \int_{\Delta X} \eta \, dP \circ \Psi^{-1}(\cdot, \mu_n) \\ &= \int_{\mathcal{F}} \eta(\Psi(f, \mu_n)) \, dP \to \int_{\mathcal{F}} \eta(\Psi(f, \mu)) \, dP = \int_{\Delta X} \eta \, d\Psi(P, \mu).\end{aligned}$$

Hence, $\mu \mapsto \Psi(P, \mu)$ is continuous. Since ΔS is compact, a set $\{\Psi(P, \mu) | \mu \in \Delta S\}$ is compact. Continuity of $\succeq$ implies that there are $\overline{P}$ and $\underline{P}$ in $\Delta(\Delta X)$ such that $\overline{P} \succeq \Psi(P, \mu) \succeq \underline{P}$ for all $\mu \in \Delta S$. Dominance implies that $\overline{P} \succeq P \succeq \underline{P}$. Continuity shows that some mixture of $\overline{P}$ and $\underline{P}$ is indifferent to P. □

So V can be extended to $\Delta\mathcal{F}$ as $V(P) = V(\overline{P})$ $(P \sim \overline{P})$. Each choice can be identified with some real function. For each $P \in \Delta\mathcal{F}$, define $\xi_P = \int_{\mathcal{F}} v(\int_S u(f) d\mu) dP(f)$ and $\Phi = \{\xi_P | P \in \Delta\mathcal{F}\}$. Define a utility functional $I : \Phi \to \mathbb{R}$ as $I(\xi) = V(P)$ $(\xi = \xi_P)$. I is well-defined by $\mathcal{U}$-dominance.

Lemma 2. 1. I is monotone; if $\xi(\mu) \geq \zeta(\mu)$ for all $\mu \in \Delta S$, then $I(\xi) \geq I(\zeta)$.
2. $I(\alpha) = \alpha$ for any $\alpha \in v(u(\Delta X))$.
3. I is positively homogeneous; $I(\alpha\xi) = \alpha I(\xi)$ for $\alpha \geq 0$.
4. If $\succeq$ satisfies First-stage independence, I is linear; $I(\alpha\xi + \zeta) = \alpha I(\xi) + I(\zeta)$.
5. If $\succeq$ satisfies Convexity and First-stage Certainty Independence, I is superlinear and satisfies C-Independence; $I(\xi + \alpha) = I(\xi) + \alpha$.
6. For any $\xi, \zeta \in \Phi$, $I(\xi) = I(\zeta)$ if $\xi(\mu) = \zeta(\mu)$ for all $\mu \in \Delta_{\mathcal{U}}$.

Proof. 1. First note that $\xi_P(\mu) = \int_{\mathcal{F}} v(\int_S u(f)d\mu)dP = \int_{\Delta X} v(u(p))d\Psi(P,\mu)$. If $\xi_P(\mu) \geq \xi_Q(\mu)$ for all $\mu \in \Delta S$, then $\Psi(P,\mu) \succeq \Psi(Q,\mu)$ for all $\mu \in \Delta S$. Dominance implies $P \succeq Q$, hence $I(\xi_P) \geq I(\xi_Q)$.
2. Easy.
3. First I show $I(\alpha\xi) = \alpha I(\xi)$ for $\alpha \geq 0$. Take $p \in \Delta X$ with $v(u(p)) = 0$. For any $P \in \Delta\mathcal{F}$,

$$\begin{aligned}\alpha\xi_P(\mu) &= \alpha\xi_P(\mu) + (1-\alpha)v(u(p)) \\ &= \alpha\int_{\mathcal{F}} v(\int_S u(f)d\mu)dP + (1-\alpha)\int_{\mathcal{F}} v(\int_S u(f)d\mu)d\delta_p \\ &= \int_{\mathcal{F}} v(\int_S u(f)d\mu)d(\alpha P + (1-\alpha)\delta_p) \\ &= \xi_{\alpha P + (1-\alpha)\delta_p}.\end{aligned}$$

Take some $\overline{P} \in \Delta(\Delta X)$ that satisfies $\overline{P} \sim P$. By First-stage independence, $\alpha P + (1-\alpha)\delta_p \sim \alpha\overline{P} + (1-\alpha)\delta_p$. From mixture linearity of V on $\Delta(\Delta X)$, $I(\alpha\xi_P) = V(\alpha P + (1-\alpha)\delta_p) = V(\alpha\overline{P} + (1-\alpha)\delta_p) = \alpha V(P) + (1-\alpha)V(\delta_p) = \alpha I(\xi_P)$.
4. First, I show additivity. Take any $P, Q \in \Delta\mathcal{F}$ and $\overline{P}, \overline{Q} \in \Delta(\Delta X)$ such that $P \sim \overline{P}$, $Q \sim \overline{Q}$. By First-stage independence $\frac{1}{2}P + \frac{1}{2}Q \sim \frac{1}{2}\overline{P} + \frac{1}{2}\overline{Q}$. Hence $V(\frac{1}{2}P + \frac{1}{2}Q) = V(\frac{1}{2}\overline{P} + \frac{1}{2}\overline{Q}) = \frac{1}{2}V(\overline{P}) + \frac{1}{2}V(\overline{Q})$. Additivity follows from positive homogeneity.

Additivity and (2) imply $I(\xi) + I(-\xi) = I(0) = 0$, hence, $I(-\xi) = -I(\xi)$. This completes the proof of linearlity.
5. First, I prove C-independence. Let $P \in \Delta\mathcal{F}$, $\overline{P}$ and $p \in \Delta X$ such that $P \sim \overline{P}$ and $v(u(p)) = \alpha$. Note that $\xi_{\overline{P}} \equiv I(\xi_P)$. $I(\xi_P + \alpha) = 2I(\frac{1}{2}\xi_P + \frac{1}{2}\xi_{\delta_p}) = 2I(\frac{1}{2}\xi_{\overline{P}} + \frac{1}{2}\xi_{\delta_p}) = I(I(\xi_P) + \alpha) = I(\xi_P) + \alpha$. The second equality follows from First-stage Certainty Independence. The last equality follows from (2).

Let $P, Q \in \Delta\mathcal{F}$ and $\alpha \in [0,1]$. I prove $I(\alpha\xi_P + (1-\alpha)\xi_Q) \geq \alpha I(\xi_P) + (1-\alpha)I(\xi_Q)$. One can assume $\xi_R \equiv \xi_Q + (I(\xi_P) - I(\xi_Q)) \in \Phi$ by homogeneity and convexity of Φ without loss of generality. From definition $I(\xi_R) = I(\xi_Q) + (I(\xi_P) - I(\xi_Q)) = I(\xi_P)$ and Convexity shows $I(\alpha\xi_P + (1-\alpha)\xi_R) = V(\alpha P + (1-\alpha)R) \geq V(P) = I(\xi_P)$. Noticing $I(\alpha\xi_P + (1-\alpha)\xi_R) = I(\alpha\xi_P + (1-\alpha)(\xi_Q + (I(\xi_P) - I(\xi_Q))))$, $I(\alpha\xi_P + (1-\alpha)\xi_Q) \geq \alpha I(\xi_P) + (1-\alpha)I(\xi_Q)$.
6. From $\mathcal{U}$-Dominance, $P \sim Q$ if $\xi_P(\mu) = \xi_Q(\mu)$ for all $\mu \in \Delta_{\mathcal{U}}$, hence, $I(\xi_P) = I(\xi_Q)$. □

Next, I characterize $\Delta_{\mathcal{U}}$.

Lemma 3. Suppose $\succeq$ satisfies Order, Continuity, Second-stage independence, $\mathcal{U}$-dominance, Nondegenerate. Then, there exists an additive set function $\mu^* : \mathcal{U} \to [0,1]$ such that $\Delta_{\mathcal{U}} = \{\mu \in \Delta S | \forall E \in \mathcal{U},\ \mu(E) = \mu^*(E)\}$.

Proof. If $\Delta_{\mathcal{U}} = \emptyset$, then $\forall P, Q \in \Delta\mathcal{F},\ P \sim Q$ by $\mathcal{U}$-dominance and this contradicts Nondegenerate. Hence $\Delta_{\mathcal{U}}$ is nonempty.

Take any expected utility function $u : \Delta X \to \mathbb{R}$ that represents $\succeq |_{\Delta X}$. For any $\mu \in \Delta_{\mathcal{U}}$, partiton $\pi \subset \mathcal{U}$, and $f \in \mathcal{F}_\pi$, define $V_\pi^\mu(f) = \sum_{E\in\pi} u(f(E))\mu(E)$. Because $f \sim \Psi(f,\mu), V_\pi^\mu(\cdot) = u \circ \Psi(\cdot,\mu)$ represents $\succeq |_{\mathcal{F}_\pi}$.

In light of uniqueness result of SEU representation, all $\mu \in \Delta_{\mathcal{U}}$ agree on $\mathcal{U}$. Therefore one can define $\mu^* : \mathcal{U} \to [0,1]$ as $\mu^*(E) = \mu_0(E)\ \forall E \in \mathcal{U}$ using any fixed μ_0. From the definition of $\Delta_{\mathcal{U}}$, $\Delta_{\mathcal{U}} \subset \{\mu \in \Delta S | \forall E \in \mathcal{U},\ \mu(E) = \mu^*(E)\}$.

In order to prove converse inclusion, take any $\mu \in \Delta S$ such that $\forall E \in \mathcal{U},\ \mu(E) = \mu^*(E)$. Take any $f \in \mathcal{F}_{\mathcal{U}}$. Then $f \in \mathcal{F}_\pi$ for some partition $\pi \subset \mathcal{U}$ and

$$V_\pi^{\mu_0}(f) = \sum_{E\in\pi} u(f(E))\mu_0(E) = \sum_{E\in\pi} u(f(E))\mu(E) = u(\Psi(f,\mu)) = V_\pi^{\mu_0}(\Psi(f,\mu)).$$

Because $V_\pi^{\mu_0}$ represents $\succeq |_{\mathcal{F}_\pi}$, $f \sim \Psi(f,\mu)$. This means $\mu \in \Delta_{\mathcal{U}}$. □

By Lemma 2 (5), preference rankings on Φ are known by values on $\Delta_{\mathcal{U}}$. Define $\Phi_0 = \{\xi|_{\Delta_{\mathcal{U}}} | \xi \in \Phi\}$ and $I_0 : \Phi_0 \to \mathbb{R}$ as $I_0(\xi_0) = I(\xi)$ if $\xi_0 = \xi|_{\Phi_0}$. Φ_0 is a subset of $C(\Delta_{\mathcal{U}})$, the set of continuous real functions on $\Delta_{\mathcal{U}}$. In the followings, $C(\Delta_{\mathcal{U}})$ is endowed with the uniform topology. Finally, main results are proven.

Proof of Theorem 1. Using linearity I_0 can be extended to $span(\Phi)$ and it is a positive linear functional. Clearly, $1 \in span(\Phi) \cap int(C(\Delta_{\mathcal{U}})_+)$. Hence it can be extended to a continuous positive linear functional $\overline{I}$ on $C(\Delta_{\mathcal{U}})$ using Ok (2007) p594 Proposition 12. Applying the Riesz representation theorem one can write $\overline{I}(\xi) = \int_{\Delta_{\mathcal{U}}} \xi dm_0$ by some $m_0 \in \Delta(\Delta_{\mathcal{U}})$. Define $m \in \Delta(\Delta S)$ as $m(B) = m_0(B \cap \Delta_{\mathcal{U}})$ for all Borel sets $B \in \mathcal{B}(\Delta S)$. Then, (u, v, m) is a SOSEU representation of $\succeq$ and $m(\Delta_{\mathcal{U}}) = 1$. This and Lemma 3 completes the proof. □

Proof of Theorem 2. Because I_0 is homogeneous of degree one and Φ_0 is convex, I_0 can be extended to $span(\Phi_0)$ as $\overline{I}_0(\alpha\xi) = \alpha I_0(\xi)$ $(\alpha \geq 0)$. Define $\hat{I} : C(\Delta_{\mathcal{U}}) \to \mathbb{R}$ by $\hat{I}(\xi) = \sup_{\xi' \in span(\Phi_0)}\{\overline{I}_0(\xi') + \inf_{\mu\in\Delta_{\mathcal{U}}}[\xi(\mu) - \xi'(\mu)]\}$. Then, $\hat{I}$ is a monotonic, superlinear and C-independent extension of I_0. Following the line of argument in the proof of Gilboa and Schmeidler (1989) Lemma 3.5 and using Aliprantis and Border (2006) Corollary 14.15, one can take a compact convex set $M_0 \subset \Delta(\Delta_{\mathcal{U}})$ of countably additive probabilities such that $\hat{I}(\xi) = \min_{m\in M_0} \int_{\Delta_{\mathcal{U}}} \xi dm$. Define $M = \{m \in \Delta(\Delta S) | m|_{\mathcal{B}(\Delta_{\mathcal{U}})} = m_0$ for some $m_0 \in M_0\}$. Then, (u, v, M) is a representation needed. □

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