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Axiomatic models of correlation misperception

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Abstract

In varioue economic enviroments, economic agents fail to correctly perceive correlations of random variables. This paper provides and axiomatizes models that formalize such agents. I introduce a new framework in which a decision maker chooses an act given an object probability over a state space. I introduce axioms that are equivalent to a representation in which his belief is a convex combination of the objective probability and prababitilies with the same marginal distributions as the objective one.

1 Introduction

In various economic environment, agents face correlations of random variables he concerens. For a typical example, investors may want to estimate covariance matrix of stocks and hedge the risk he takes. In order to understand such behavior, it is fundamental to understand how people process correlating informations.

While standard models of decision making assume that agents are smart enough to perfectly understand joint distributions of economic variables, recent laboratry experiments show that people often misunderstand correlations. For example, Eyster and Weizsacker (2016) shows that most people neglect correlation of assets when they form a portfolio. Enke and Zimmermann (2017) shows that when people updates their beliefs from multiple information sources, they tend to neglect their correlations.

In order to accomodate the experimental findings above, this paper provides and axiomatizes two models of decision maker who may misunderstand correation structures of economic variables. As a result, we can distinguish agents who misunderstand information from those do not. Two models we consider is different in the modes of misunderstandings they capture. While one model represents DM who underestimates the correlations and is ignorant of own misperception, another model represents DM who recognizes the possibility of own misperception.

In the framework, the decision maker (DM) faces a state space with Cartesian product structure. DM is given an objective probability over the state space. One can interpret this probability as one given by an experimenter in laboratry settings, or as one DM would have estimated with an econometric technique in financial settings. DM may or may not correctly process this probability. Without knowing the state, but with the objective probability of the states, he chooses an act and then obtain a state dependent outcome. Introducing objective probabilities enables us to distinguish DM who misunderstand correlations and those do not.

I compare paper with existing decsion theoretic work. Ellis and Piccione (2017) is the first axiomatic paper on correlation misperception. They introduced a new framework and analyzed a model that resembles subjective expected utility model. This paper is different from theirs in that it introduces object probabilities. As I noted above, this component is indispensable to distinguish DM with and without correlation misperception.

My models resembles well known ϵ-contaminated model that represent beliefs of DM under uncertainty as a convex combination of a set of probabilities and a single probability. It is axiomatized by Nishimura and Ozaki (2006), Gajdos et al. (2008), and Kopylov (2009). The most related paper is Gajdos et al. (2008). They use a framework with objective, but ambiguous information. While the center of the belif is endogenously choosen in their model, it is exogenously given in this paper.

Rest of the paper is as follows. Section 2 introduces the setup and the models. Section 3 provides the axioms and state the representation theorem. Proofs are in appendix.

2 Models of correlation misperception

This section introduces two models of correlation misperception. In order to represent choce behavior that is driven by misperception, we need a framework that incorporates objective probability distribution.

2.1 Setup

Let finite sets Ω_i $(i = 1, \dots n)$ be the different information sources. Define $\Omega = \prod_{i=1}^{n} \Omega_i$ and interpret it as the state space. The set of probabilities over Ω is written as $\Delta\Omega$. A convex subset X of a linear space is called the set of outcomes. A function $f : \Omega \to X$ is called an act and let $\mathcal{F}$ be the set of all acts.

The choice set of DM is $\mathcal{C} = \mathcal{F} \times \Delta\Omega$. When an alternative (f, p) is provided to DM, the objective probability distribution of the state is p, and DM obtain $f(\omega)$ if ω obtains. Our data of DM's choice behavior is a binary relation $\succeq$ over $\mathcal{C}$.

2.2 Model of correlation underestimation

We consider a model of correlation underestimation. It is a model of DM who underestimates the degree of correlation of information sources and, in addition, he is entirely unaware of it.

Suppose that such DM is told that the objective probability is $p \in \Delta\Omega_1 \times \Omega_2$. Even if he can not perceive the true correlation, he would correctly understand the marginal distributions p_1 and p_2 over Ω_1 and Ω_2. What happens if he can not understand correlation at all? In this case, realizations of $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ is independent according to his belief. Then, his subjective probability over $\Delta\Omega$, if any, must be the product probability of p_1 and p_2.

The degree of correlation underestimation should be different from one another. So it is worth to consider intermediate correlation underestimation, in addition to the two extreme cases of correct understanding and full neglect. I model such cases as if DM has a subjective probability that is a mixture of the objective probability and the product probability of its marginals.

For any $p \in \Delta\Omega$, define $[p] \in \Delta\Omega$ as

$$[p](\omega_1, \ldots, \omega_n) = \prod_{i=1}^{n} p_i(\omega_i),$$

where p_i the marginal probability of p over Ω_i. A correlation underestimation representation of $\succeq$ is a pair (u, ϵ) of affine function $u : X \to \mathbb{R}$ and a number $\epsilon \in [0, 1]$ such that $\succeq$ is represented by

$$\begin{aligned} V(f, p) &= \int_\Omega u(f) \mathrm{d}\pi_p, \\ \pi_p &= (1 - \epsilon)p + \epsilon[p]. \end{aligned}$$

2.3 Model of correlation uncertainty

Even if DM can not perceive the objective probability correctly, he may not just underestimate correlation. It is possible that DM is aware of his ignorance of correlation. I call such situations as correlation ignorance.

For $p \in \Delta\Omega$, define $M[p] = \{\pi \in \Delta\Omega | \ [\pi] = [p]\}$. That is, $M[p]$ is the set of probabilities that have the same marginal probabilities as p. Given the objective probability p, DM correctly recognizes that $M[p]$ includes the true probability law, even if he does not know p is the true one.

Such DM is described by the utility function below.

$$\begin{aligned} V(f, p) &= \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi, \\ \Pi_p &= (1 - \epsilon)p + \epsilon M[p]. \end{aligned}$$

The parameters of this utility function is an affine function $u : X \to \mathbb{R}$ and a number $\epsilon \in [0, 1]$. This is a special case of ϵ-contamination model.

3 Axiomatic foundation

This section provide axiomatic foundations of the two models presented in the last section. Knowing their differences in terms of behavior, we can directly test which model more accurately approximate choice behavior observed in labolatory. To this end, we first introduce a general model that include the two as special cases, and axiomatize it. By doing so, we can treat two models from unified viewpoint.

For each $p \in \Delta\Omega$, let $\Delta_p \subset \Delta\Omega$ be the set of probabilities that contaminates the belief of DM. Let $\Delta = \{\Delta_p\}_{p \in \Delta\Omega}$ be a family of such sets. In what follows, we consider a model

$$\begin{aligned} V(f, p) &= \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi, \\ \Pi_p &= (1 - \epsilon)p + \epsilon\Delta_p. \end{aligned}$$

When we set $\Delta_p = [p]$, we obtain correlation underestimation model. On the other hand, setting $\Delta_p = M[p]$, we obtain correlation uncertainty model. Henceforth we assume $\Delta_p \neq \{p\}$ for some p.

Axiom 1 (Order)**.** *$\succeq$ is complete, transitive, and nondegenerate.*

Axiom 2 (Continuity)**.** *$(f,p) \succ (g,p) \succ (h,p)$ implies that there exists $\alpha, \beta \in (0,1)$ such that $(\alpha f + (1-\alpha)h, p) \succ (g,p) \succ (\beta f + (1-\beta)h, p)$.*

Axiom 3 (Risk Preference)**.** *For any $x \in X$ and $p, q \in \Delta\Omega$, $(x,p) \sim (x,q)$ holds.*

Axiom 4 (Monotonicity)**.** *If $(f(\omega),p) \succeq (g(\omega),p)$ for all $\omega \in \Omega$, then, $(f,p) \succeq (g,p)$.*

Axiom 5 (Certainty Independence)**.** *If $(f,p) \succeq (g,p)$, $x \in X$, and $\alpha \in [0,1]$, $(\alpha f + (1-\alpha)x, p) \succeq (\alpha g + (1-\alpha)x, p)$.*

Axiom 6 (Uncertainty Aversion)**.** *If $(f,p) \succeq (g,p)$ and $\alpha \in [0,1]$, $(\alpha f + (1-\alpha)g, p) \succeq (g,p)$.*

Let

$$L(f,p) = \{x \in X | \forall q \in \Delta_p \ (f,q) \succeq (x,q)\}.$$

Axiom 7 (Δ-Dominance)**.** *If $(\langle f \rangle_p, p) \succeq (\langle g \rangle_q, q)$ and $L(f,p) \supseteq L(g,q)$, then $(f,p) \succeq (g,q)$.*

Theorem 1. *The followings are equivalent.*

1. *$\succeq$ satisfies axioms (1)-(7).*
2. *There exists an affinie function $u : X \to \mathbb{R}$ and $\epsilon \in [0,1]$ such that the utility function defined as*

$$V(f,p) = \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi,$$
$$\Pi_p = (1-\epsilon)p + \epsilon \Delta_p.$$

represents $\succeq$.

Adding next two axioms turns general correlation misperception model into correlation underestimate model.

Axiom 8 (Dominance)**.** *If $(\langle f \rangle_p, p) \succeq (\langle g \rangle_q, q)$ and $([f]_p, p) \succeq ([g]_q, q)$, then $(f,p) \succeq (g,q)$.*

Proposition 1. *The following are equivalent.*
1. *$\succeq$ satisfies axioms (1)-(6),(8).*
2. *There exists an affinie function $u : X \to \mathbb{R}$ and $\epsilon \in [0,1]$ such that the utility function defined as*

$$V(f,p) = \int_\Omega u(f) \mathrm{d}\pi_p,$$
$$\pi_p = (1-\epsilon)p + \epsilon[p].$$

Next, we turn to axiomatization of Correlation Uncertainty Representation. Let $L_M(f,p) = \{x \in X | \forall q \in M[p] \ (f,q) \succeq (x,q)\}$.

Axiom 9 (Worst Dominance)**.** *If $(\langle f \rangle_p, p) \succeq (\langle g \rangle_q, q)$ and $L_M(f,p) \supseteq L_M(g,q)$, then $(f,p) \succeq (g,q)$.*

Proposition 2. *The followings are equivalent.*
1. $\succeq$ *satisfies axioms (1)-(6),(9).*
2. *There exists an affinie function* $u : X \to \mathbb{R}$ *and* $\epsilon \in [0,1]$ *such that the utility function defined as*

$$V(f,p) = \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi,$$
$$\Pi_p = (1-\epsilon)p + \epsilon M[p].$$

A Appendix

A.1 Proof of Theorem 1

For each $p \in \Delta\Omega$, define a binary relation $\succeq_p$ over $\mathcal{F}$ as

$$f \succeq_p g \Leftrightarrow (f,p) \succeq (g,p).$$

Lemma 1. *If* $\succeq$ *satisfies Order, Continuity, Risk Preference, Monotonicity, Uncertainty Aversion,and Certainty Independence. Then, the followings hold.*

1. *There exist an affine function* $u : X \to \mathbb{R}$ *and a family of convex sets of probabilities* $\{\Pi_p\}_{p \in \Delta\Omega}$ *such that* $\succeq_p$ *is represented by the utility function*

$$U_p(f,p) = \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi.$$

2. *There is a function* $V : \mathcal{C} \to \mathbb{R}$ *with a connected range that represents* $\succeq$.

Proof. By the assumption, $\succeq_p$ has a maxmin representation. That is, there exists a affine function $u_p : X \to \mathbb{R}$ and a closed convex set $\Pi_p \subseteq \Delta\Omega$ so that the function

$$U_p(f) = \min_{\pi \in \Pi_p} \int_\Omega u_p(f) \mathrm{d}\pi$$

represents $\succeq_p$.

Fix some $p^* \in \Delta\Omega$. For any $q \in \Delta\Omega$ and $x, y \in X$, by axiom 3, $x \succeq_{p^*} y \Leftrightarrow x \succeq_q y$. Because u_{p^*} and u_q are affine functions representing a common preference relation over X, u_q is an affine transform of u_{p^*}. So without loss of generality, normalize as $u_q = u_{p^*}$ and write u_{p^*} just as u.

Next, we construct certainty equivalents of choices. Take any $(f,p) \in \mathcal{C}$ and let $\overline{x}, \underline{x} \in X$ be outcomes such that

$$(\overline{x}, p) \succeq (f,p) \succeq (\underline{x}, p).$$

Such $\overline{x}$ and $\underline{x}$ exist because $\succeq$ satisfies Monotonicity. If $(x,p) \sim (f,p)$ holds for $x \in \{\overline{x}, \underline{x}\}$, then set $c(f,p) = x$. Otherwise, $(\overline{x}, p) \succ (f,p) \succ (\underline{x}, p)$ holds. Because of Continuity and Certainty Independence, the sets

$$\{\alpha \in [0,1] | \ (\alpha\overline{x} + (1-\alpha)\underline{x}, p) \succeq (f,p)\}, \ \{\alpha \in [0,1] | \ (f,p) \succeq (\alpha\overline{x} + (1-\alpha)\underline{x}, p)\}$$

are closed. Since they cover a connected set $[0,1]$, they have nonempty intersection. Hence, there is some $\alpha \in [0,1]$ such that $(f,p) \sim (\alpha\overline{x} + (1-\alpha)\underline{x}, p)$. Set $c(f,p) =$

$\alpha\overline{x} + (1-\alpha)\underline{x}$. We defined a function $c : \mathcal{C} \to X$ with a property $(f,p) \sim (c(f,p),p)$. By the construction, V has a connected range.

Define $V : \mathcal{C} \to \mathbb{R}$ as $V(f,p) = u(c(f,p))$. Since $\succeq$ satisfies Risk Preference, $(x,p) \sim (x,q)$ holds for any $x \in X$ and $p, q \in \Delta\Omega$. Hence,

$$(f,p) \succeq (g,q) \Leftrightarrow (c(f,p),p) \succeq (c(g,q),q) \Leftrightarrow V(f,p) \geq V(g,q).$$

The function V represents $\succeq$. $\square$

It is without generality to, and I do, assume $(-1,1) \subset u(X)$ in what follows.

I adopt the following characterization result of the arithmetic mean to axiomatize the two representations in the text.

Theorem 2 (Aczél (1966) p. 234)**.**

$$W = (1-q)x + qy$$

is the most general function of two variables satisfying the pair of functional equations

$$W(x+t, y+t) = W(x,y) + t, \tag{1}$$

$$W(\alpha x, \alpha y) = \alpha W(x,y) \;\; \alpha \neq 0 \tag{2}$$

Suppose that $\succeq$ satisfies $\succeq$ satisfies axioms Order, Continuity, Risk Preference, Monotonicity, Independence, Dominance.

By the Lemma 1, $\succeq_p$ has the representation

$$U_p(f,p) = \min_{\pi \in \Pi_p} \int_\Omega u(f) \mathrm{d}\pi. \tag{3}$$

Let

$$\mathcal{U} = \left\{ \left(\int u(f)dp, \min_{\pi \in \Delta_p} \int u(f) d\pi \right) \mid (f,p) \in \mathcal{C} \right\}.$$

Since $(x,p) \succeq (y,q) \Leftrightarrow u(x) \geq u(y)$, Δ-Dominance implies that if $\int u(f)dp = \int u(g)dq$ and $\min_{\pi \in \Delta_p} \int u(f)d\pi = \min_{\pi \in \Delta_p} \int u(g)d\pi$, then $V(f,p) = V(g,q)$. So there exists a function $W : \mathcal{U} \to \mathbb{R}$ such that

$$V(f,p) = W\left(\int u(f)dp, \min_{\pi \in \Delta_p} \int u(f)d\pi \right). \tag{4}$$

Over the subdomain $\mathcal{C}_p = \{(f,p) \in \mathcal{C} |\; f \in \mathcal{F}\}$, U_p and V represents the same preference. Thus, transforming V appropriately, we have

$$V(f,p) = W\left(\int u(f)dp, \int u(f) d[p] \right) = U_p(f,p).$$

Because of the representation (3), W satisfies (1) and (2) if $(x,y), (x+t, y+t), (\alpha x, \alpha y) \in \mathcal{C}$.

Let $\mathcal{U}^* = \{(x,y) \in \mathbb{R}^2 | x \geq y\}$. We shall extend W to $\mathcal{U}$.

Lemma 2. *If $\Delta_p \neq \{p\}$, for any $(\alpha, \beta) \in \mathcal{U}^*$, there exists $f \in \mathcal{F}$ and $\epsilon > 0$ so that*

$$\epsilon \left(\int u(f)dp, \min_{\pi \in \Delta_p} \int u(f)d\pi \right) = (\alpha, \beta).$$

Proof. Take any $(\alpha, \beta) \in \mathcal{U}^*$. Then, there exists $\omega^* \in \Omega$ and $q \in \Delta_p \backslash \{p\}$ such that

$$q(\omega^*) = \min_{\pi \in \Delta_p} \pi(\omega^*),$$
$$\delta = p(\omega^*) - q(\omega^*) > 0.$$

Let $\gamma = q(\omega^*)$ and note $\gamma < 1$.

For $\alpha_0 \geq 0$, consider an act so that

$$u \circ f(\omega) = \begin{cases} \alpha_0 & (\omega = \omega^*) \\ \beta_0 & (\omega \neq \omega^*). \end{cases} \tag{5}$$

If such an act exists, then

$$\int u(f) dp = (\delta + \gamma)\alpha_0 + (1 - \gamma - \delta)\beta_0,$$
$$\int u(f) dq = \gamma\alpha_0 + (1 - \gamma)\beta_0.$$

We will find $(\alpha_0,\ \beta_0)$ such that $\int u(f)dp = \epsilon\alpha$ and $\int u(f)d[p] = \epsilon\beta$. Consider a linear equation

$$\begin{pmatrix} \delta + \gamma & 1 - \gamma - \delta \\ \gamma & 1 - \gamma \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Because the 2×2 matrix on the left hand side is regular, the equation has a solution $(\alpha_0,\ \beta_0)$. Multiplying both hands by $(1, -1)$, we note $\alpha_0 \geq \beta_0$. And (α_0, β_0) converges to zero as $\epsilon \to 0$. Since $(-1, 1) \subset u(X)$, for sufficiently small $\epsilon > 0$, there exists an act that satisfies (5). $\square$

Extend W to $\mathcal{U}^*$ as

$$W^*(\alpha, \beta) = \lambda W(\alpha_0, \beta_0)$$

for $(\alpha_0, \beta_0) \in \mathcal{U}^*$ and $\lambda \geq 0$ such that $(\alpha, \beta) = \lambda(\alpha_0, \beta_0)$. This extension is well-defined because of the representation (3). Then, further extend W^* to $\mathbb{R}^2$ as

$$W^{**}(\alpha, \beta) = \begin{cases} W^*(\alpha, \beta) & \text{if } (\alpha, \beta) \in \mathcal{U}^*, \\ -W^*(-\alpha, -\beta) & \text{if } (\alpha, \beta) \notin \mathcal{U}^*. \end{cases}$$

One can show that W^{**} satisfies equations (1) and (2). Then, applying Theorem 2, we obtain $\epsilon \in \mathbb{R}$ such that

$$W(\alpha, \beta) = (1 - \epsilon)\alpha + \epsilon\beta.$$

From Monotonicity, $\epsilon \in [0, 1]$. Then from (4),

$$V(f, p) = (1 - \epsilon) \int u(f) dp + \epsilon \min_{\pi \in \Delta_p} \int u(f) d\pi.$$

$\square$

A.2 Proof of Propositions

First, we show Proposition 1. Let $\Delta_p = [p]$ and thus $L(f,p) = \{x \in X | (f,[p]) \succeq (x,[p])\}$. What we show that $\succeq$ satisfies Δ-Dominance. To do so, it is sufficient to prove,

$$L(f,p) \supset L(g,q) \Rightarrow (f,[p]) \succeq (g,[q]).$$

Suppose $(g,[q]) \succ (f,[p])$. Because V has a convex range, there exists some $x \in X$ such that

$$(g,[q]) \succ (x,[q]) \sim (x,[p]) \succ (f,[p]).$$

This contradicts $L(f,p) \supset L(g,q)$.

Proposition 2 is a direct implication of Theorem 1. $\square$

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